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TOPOLOGICAL SUBGRAPHS IN GRAPHS OF LARGE GIRTH

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It is proved that for every finite graph H of maximum degree $n \geq 3$, there is an integer $g(H)$ such that every finite graph of minimum degree n and girth at least $g(H)$ contains a subdivision of H. This had been conjectured for $H = K_{n+1}$ in [8]. We prove also that every finite $2n$-connected graph of sufficiently large girth is n-linked, and this is best possible for all $n \geq 2$.

1. Introduction

It was proved in [6] that every finite graph G (without loops and without multiple edges) with minimum degree $\delta(G)$ large enough (dependent only on n) contains a $\dot{K}_{n+1}$, where $\dot{K}_n$ denotes any subdivision of the complete graph K_n. In general, one has to take $\delta(G)$ much larger than n to insure the existence of a $\dot{K}_{n+1}$: there are graphs G with $\delta(G) \geq \frac{1}{8}n^2$ for $n \geq 2$, but without a $\dot{K}_{n+1}$ (cf. [3]). But we will prove in this paper that *$\delta(G) \geq n$ suffices for the existence of a $\dot{K}_{n+1}$ in G, if the girth $\tau(G)$ of G is large enough.* (Note that there are finite graphs of arbitrarily large minimum degree and arbitrarily large girth by [12] and [9], for instance.) This was conjectured in [8], and is related to a result of C. Thomassen [11] which says that a graph G with $\delta(G) \geq 3$ and $\tau(G)$ large enough contains K_n as a minor. We will even prove the following stronger result, where $\Delta(H)$ denotes the maximum degree of the graph H: *For every finite graph H, there is an integer $g(H)$ such that every finite graph G with $\delta(G) \geq max\{\Delta(H), 3\}$ and girth $\tau(G) \geq g(H)$ contains a subdivision $\dot{H}$ of H.*

We need some further notation. By a *graph* we mean always a simple graph, i.e. a graph without loops or multiple edges. *All graphs are assumed to be finite, unless otherwise specified.* The vertex number and the edge number of a graph G are denoted by $|G|$ and $||G||$, respectively, the edge joining the vertices x and y by $[x,y]$. For a graph $G, x \in G$ always means $x \in V(G)$ and $G(A)$ is the subgraph of G induced

Mathematics Subject Classification (1991): 05C35 by $A \subseteq V(G)$. The distance between the vertices x and y in a graph G is denoted by $d_G(x,y)$. For a non-negative integer n and an $x \in G, B_n(x) := \{y \in G : d_G(y,x) = n\}$ and $\overline{B}_n(x) := G(\bigcup_{m=0}^{n} B_m(x))$. For $x \in G$, $N_G(x) := B_1(x)$, and for $A \subseteq V(G)$ and a subgraph C of $G, N_G(A) := \bigcup_{x \in A} N_G(x) - A$ and $N_G(C) := N_G(V(C))$, respectively. A path with endvertices x and y is called an *x,y-path* or *(x,y)-path*. For vertices x and y on a path $P, P[x,y]$ denotes the x,y-subpath contained in P. A graph G is *n-linked* for a positive integer n, if $|G| \geq 2n$ and for every sequence $a_1, a_2, \ldots, a_{2n}$ of distinct vertices of G, there are (pairwise vertex-) disjoint a_i, a_{i+n}-paths in G for $i = 1, \ldots, n$. The (vertex-) connectivity number of a graph G is denoted by $\kappa(G)$. For $x \in G$ and $A \subseteq V(G - x)$, an *x,A-path* is an x,a-path P with $V(P) \cap A = \{a\}$. A set of x,A-paths $P_1 \ldots, P_n$ with $V(P_i) \cap V(P_j) = \{x\}$ for all $i \neq j$ is called an *x,A-fan of order n ending in $A \cap (\bigcup_{i=1}^{n} V(P_i))$*. For a set A and an integer n, $\mathcal{P}_n(A) := \{A' \subseteq A; |A'| = n\}$ and $\mathbb{N}_n := \{1, \ldots, n\}$.

We will apply the following result, proved by Jung and Larman/Mani, independently, in 1970.

Theorem A. ([4] and [5]) *For every positive integer n, there is a least integer $f(n)$ such that every finite $f(n)$-connected graph is n-linked.*

We will study in section 3 the dependence of the function f on the girth. It is easily seen and well known that every n-linked graph is $(2n-1)$- connected (cf. [1], for instance). If G arises from K_{3n-1} by deleting n independent edges, $\kappa(G) = 3n-3$ holds, but G is not n-linked. On the other side, it is proved in [1] that every $(22n)$-connected graph is n-linked. We will show that *$\kappa(G) \geq 2n$ and $\tau(G)$ large enough imply that G is n-linked, but $\kappa(G) \geq 2n-1$ does not.* This is similar to a result of C. Thomassen [10] that an uncountable $2n$-connected graph is n-linked, but a $(2n-1)$-connected one, in general, is not.

2. Subdivisions of a given graph in graphs of large girth

In this section we will prove the main result of the paper, that a subdivision of an arbitrary connected graph H is contained in every graph G with $\delta(G) \geq \Delta(H)$ and $\tau(G)$ large enough. We give a detailed proof for the case $H = K_{n+1}$ and point then out the straightforward modifications for the general case.

Theorem 1. *For every positive integer n, there is an integer $g(n)$ such that every finite graph G with $\delta(G) \geq n$ and $\tau(G) \geq g(n)$ contains a subdivision of K_{n+1}.*

Proof. We assume $n \geq 3$. Define $c_0 := 3(f(\binom{n+1}{2}) + n)$, where f is the function from Theorem A, and choose a positive integer d_0 with $n(n-1)^{d_0} \geq c_0$.

Let G be a graph with $\delta(G) \geq n$ and $t_0 := \tau(G) \geq 2(4d_0+1)(c_0+3n+1)$. We show that G contains a $\dot{K}_{n+1}$.

Choose X_0 maximal in $\{X \subseteq V(G) : d_G(x,y) > 2d_0$ for all $\{x,y\} \in \mathcal{P}_2(X)\}$. Then the balls $\overline{B}_{d_0}(x)$ are disjoint for $x \in X_0$ and for every $y \in G$ there is an $x \in X_0$

with $d_G(y,x) \le 2d_0$. First distribute the vertices of distance d_0+1 from X_0 in G to the $\overline{B}_{d_0}(x)$ $(x \in X_0)$ so that the induced subgraphs remain connected, then the vertices of distance d_0+2 from X_0, and so on. Continuing in this way, in d_0 steps, we get connected induced subgraphs T_x of G with $\overline{B}_{d_0}(x) \subseteq T_x \subseteq \overline{B}_{2d_0}(x)$ for $x \in X_0$ such that $V(T_x)(x \in X_0)$ form a partition of $V(G)$. Since $t_0 > 4d_0+1$, all $\overline{B}_{2d_0}(x)$ are trees. So T_x is a tree of diameter at most $4d_0$ for every $x \in X_0$.

Let H arise from G by contracting T_x to the vertex x for every $x \in X_0$ and deleting loops. Since $t_0 > 8d_0+2$, no multiple edges are generated by the contraction. Furthermore, $\delta(H) \ge n(n-1)^{d_0} \ge c_0$, since $\overline{B}_{d_0}(x)$ has at least $n(n-1)^{d_0-1}$ endvertices, hence T_x, too. Since the diameter of T_x is at most $4d_0$, we have $\tau(H) \ge \frac{t_0}{4d_0+1} \ge 2(c_0+3n+1)$.

Choose A in $\{A' : \emptyset \ne A' \subseteq V(H) \text{ and } |N_H(A)| < c_0\}$ as small as possible. Then $|A| > c_0$, since $\delta(H) \ge c_0$ and the *graph* H has no triangle. Since $|N_H(A')| \ge c_0$ for every $\emptyset \ne A' \subsetneqq A$ by the choice of A, by a well known variation of Menger's Theorem (see, for instance, Theorem 2.3.1 in [2]), one can find pairwise disjoint edges $e_b := [b, q(b)] \in E(H)$ for $b \in B := N_H(A)$ with $q(b) \in A$, since $|A| \ge c_0 \ge |B|$. Set $Q := \{q(b) : b \in B\}$.

Let F arise from $H(A \cup B)$ by contracting the edge e_b to the vertex $q(b) \in A$ for every $b \in B$ and deleting all loops. Then $\tau(F) \ge \frac{\tau(H)}{2} \ge c_0+3n+1$, in particular, there are no multiple edges in F, and $\delta(F) \ge c_0$, since $|N_H(b) \cap A| \ge 2$ for $b \in B$ by the choice of A.

Assume $S \subseteq V(F)$ separates F. Then there is an component C of $F-S$ with $|C \cap Q| \le \frac{|Q-S|}{2}$. Returning to H by splitting the vertices $q(b) \in F$ into b and $q(b) \in H$, one gets $|N_H(V(C))| \le \frac{|Q-S|}{2} + |S| + |S \cap Q| = \frac{|Q|}{2} + |S| + \frac{|S \cap Q|}{2} < \frac{c_0}{2} + \frac{3}{2}|S|$. But this implies $|S| > \frac{c_0}{3} = f(\binom{n+1}{2}) + n$, since $|N_H(V(C))| \ge c_0$ by the choice of A and $V(C) \subsetneqq A$. Hence the connectivity number $\kappa(F) \ge f(\binom{n+1}{2}) + n + 1$, since $|F| > c_0$.

Let C_0 be a circuit in F with $|C_0| = \tau(F) \ge c_0 + 3n + 1$. It is not difficult to check, that there is a set $Y_0 \in \mathcal{P}_{n+1}(V(C_0) - Q)$ such that $d_{C_0}(y,z) \ge 3$ for all $\{y,z\} \in \mathcal{P}_2(Y_0)$. (If $Q \cap V(C_0) \ne \emptyset$, choose $y \in C_0 - Q$ with $N_{C_0}(y) \cap Q \ne \emptyset$ and consider the path $C_0 - (\{y\} \cup N_{C_0}(y))$ which has at least $3n$ vertices not in Q.) For every $y \in Y_0 \subseteq A \subseteq X_0$, there is a $y, N_G(T_y)$ - fan F_y of order n in the tree $G(V(T_y) \cup N_G(T_y))$, ending in $N_y \in \mathcal{P}_n(N_G(T_y))$, say, since $\tau(G) > 4d_0+2$. Since no multiple edges are generated by the contraction of the T_x for $x \in X_0$ and the edges e_b for $b \in B$, no two vertices of N_y are identified by constructing F. So there corresponds an $N''_y \in \mathcal{P}_n(N_H(y))$ in H to N_y. Since $y \in A$, we have $N''_y \subseteq A \cup B$, and since $y \notin Q$, no element of N''_y is identified with y constructing F from H. So there corresponds an $N'_y \in \mathcal{P}_n(N_F(y))$ in F to N_y for every $y \in Y_0$. But $d_{C_0}(y,z) \ge 3$

implies $d_F(y,z)\geq 3$, since $|C_0|=\tau(F)$, and hence $(N'_y\cup\{y\})\cap(N'_z\cup\{z\})=\emptyset$ for all $\{y,z\}\in\mathcal{P}_2(Y_0)$. We can pair the $(n+1)n$ vertices of $\bigcup_{y\in Y_0} N'_y$ in such a way into $\binom{n+1}{2}$ disjoint unordered pairs that for every $\{y,z\}\in\mathcal{P}_2(Y_0)$ there is exactly one pair $p'_{yz}=p'_{zy}$ with one vertex in N'_y, the other in N'_z. Let $p_{yz}=p_{zy}$ be the corresponding pairing for the $(n+1)n$ vertices in $\bigcup_{y\in Y_0} N_y$. Since $\kappa(F-Y_0)\geq f(\binom{n+1}{2})$, there are $\binom{n+1}{2}$ disjoint paths $P'_{yz}=P'_{zy}$ in $F-Y_0$ for $\{y,z\}\in\mathcal{P}_2(Y_0)$, where P'_{yz} is a p'_{yz}- path. Since F arises from G by identifying connected subgraphs and deleting vertices, we can replace the vertices of these paths P'_{yz} with paths in these identified subgraphs to get $\binom{n+1}{2}$ disjoint p_{yz}- paths $P_{yz}=P_{zy}$ in $G-\bigcup_{y\in Y_0} V(T_y)$ for $\{y,z\}\in\mathcal{P}_2(Y_0)$. These p_{yz}-paths P_{yz} for $\{y,z\}\in\mathcal{P}_2(Y_0)$ together with the fans F_y for $y\in Y_0$ form a subdivision of K_{n+1} in G. ∎

Remark. In our proof we have not taken much care of the bound for the necessary girth $\tau(G)$, since I believe that the use of the function f necessarily gives a much too large bound. One can avoid the conditions $d_{C_0}(y,z)\geq 3$ and $Y_0\cap Q=\emptyset$ in the last paragraph of the proof, to get a better bound. We will show now that $\tau(G)\geq(4d_0+1)(n+4)$ suffices, where d_0 is determined as above with respect to the less number $c_0:=3(f(\binom{n}{2})+n)$. Then $\tau(H)\geq n+4$. I will sketch the modifications in the proof, all concerning only the last paragraph. There is a shortest circuit C_0 in $H(A)$. Let $P: y_0,y_1,\ldots,y_n$ be a subpath of C_0 of length n and define $Y_0:=V(P)$. Since we can use every vertex $y'\in T_y$ as branch vertex of the fan F_y, we can prescribe three vertices of $N_y\subseteq N_G(T_y)$. So we can choose $N_P(y)\subseteq N'_y$ for $y\in Y_0$ and use the n edges of P for the subdivision of K_{n+1} wanted, so that we need only $\binom{n}{2}$ further junctions. Assume $y\in Y_0\cap Q$, say $y=q(b)$. Then there is exactly one edge between T_y and T_b in G, say, adjacent to $b'\in T_b$. It could happen that necessarily $b'\in N_y$, hence $b\in N''_y$. Then we can extent the fan F_y through T_b to a vertex of $N_G(T_b)\cap T_a$ for an $a\in A-\{y\}$ and replace $y\in N'_y$ with $a\in N_F(y)$. We use the same notation F_y for the extented fan and N'_y for the new set from $\mathcal{P}_n(N_F(y))$. Since $\tau(H)>n+2$, we have $N'_y\cap V(P)=N_p(y)$ for all $y\in Y_0$. If $\emptyset\neq N'_{y_i}\cap N'_{y_j}\neq\{y_{i+1}\}$, for $0\leq i<j\leq n$, then there is a circuit of length at most $j-i+4$ in H. So this can occur only for $i=0$ and $j=n$, since $\tau(H)\geq n+4$, and then $N'_{y_0}\cap N'_{y_n}=\{z\}\subseteq A-V(P)$. Then we can link F_{y_0} and F_{y_n} over T_z. All the other connecting paths of a subdivision of K_{n+1} are constructed as above. ∎

Theorem 1 implies that *every graph G with edge number $||G||\geq(n-1)|G|>0$ and large $\tau(G)$ contains a $\dot{K}_{n+1}$.* (For a minimal subgraph $H\subseteq G$ satisfying the condition $||H||\geq(n-1)|H|>0$ one has $\delta(H)\geq n$.) It was conjectured in [8] that this holds even for $||G||\geq\frac{n}{2}|G|$. This conjecture seems probable and was proved for $n=3$ in [8], but remains open for $n\geq 4$. I am convinced that one can proof the case $n=4$ by methods as above in a more sophisticated way, but at the moment

I have not much hope for the general case.* Perhaps even the following stronger result holds:

For every integer $n \geq 4$, *every finite graph* G *with* $||G|| > \frac{n}{2}(|G| - (n-1))$ *and* $n+2 \leq \tau(G) < \infty$ *contains a* $\dot{K}_{n+1}$.

For $n = 3$, it was proved in [8] that $||G|| \geq \frac{3}{2}|G| - 2, |G| \geq 4$, and $\tau(G) \geq 5$ imply the existence of a $\dot{K}_4$, and that there are infinitely many graphs G with $||G|| = \frac{3}{2}|G| - \frac{5}{2}$ and $\tau(G) = 5$, but without a $\dot{K}_4$. For $n = 4$, I know only one graph H with $||H|| = 2|H| - 6$ and $6 \leq \tau(H) < \infty$, but without a $\dot{K}_5$. We get this graph H from the graph F in Figure 6 in [8] by subdividing the four horizontal lines (i.e. the edges between vertices of degree 4) by exactly one vertex a_i for $i = 1, \dots, 4$ and by adding a further vertex b which is joined by an edge exactly to the vertices a_1, a_2, a_3, a_4. But one can easily construct infinitely many graphs G with $||G|| = 2|G| - 7$ and $\tau(G) = 6$ not containing a $\dot{K}_5$.

Let us now consider any finite graph H with $\Delta(H) \leq n$. If we define $c_0 := 3(f(||H||) + |H| - 1$ (with $f(0) := 0$), then the proof above shows that every graph G with $\delta(G) \geq n \geq 3$ and $\tau(G) \geq 2(4d_0 + 1)(c_0 + 3|H| - 2)$ has an $\dot{H}$, because then $\kappa(F) \geq f(||H||) + |H|$ and we can take $Y_0 \in \mathcal{P}_{|H|}(V(C_0) - Q)$ in the last paragraph of the proof for the branch vertices of an $\dot{H} \subseteq G$. So we have even got the following result.

Theorem 2. *For every graph* H, *there is an integer* $g(H)$ *such that every graph* G *with* $\delta(G) \geq max\{\Delta(H), 3\}$ *and* $\tau(G) \geq g(H)$ *contains a subdivision of* H.

If H is connected, we can, obviously, replace $max\{\Delta(H), 3\}$ with $\Delta(H)$. Since every complete graph is a contraction of a 3-regular graph, Theorem 2 implies that every graph G with $\delta(G) \geq 3$ and $\tau(G)$ large enough has K_n as a minor, one of the results proved by C. Thomassen (with a better bound for the necessary girth) in [11].

3. Subdivisions with prescribed branch vertices

Let H be a graph with $n := \Delta(H) \geq 3$ and $b \in H$. Then for n-connected graphs one can improve Theorem 2 in the following way: *In every* n*-connected graph* G *of sufficiently large girth, for every vertex* $a \in G$, *there is an* $\dot{H} \subseteq G$, *where* a *is the branch vertex of* $\dot{H}$ *which corresponds to* b.

The exact proof of this assertion is somewhat laborious, so I will give here only an idea: Choose any $(n-1)$-regular graph F with $\kappa(F) = n-1$ and $m := |F| \geq |H| + n$. Take $m+1$ disjoint copies $F_1, \dots, F_{m+1}$ of F and join F_i and F_j by exactly one edge

* Added in proof. In the meantime, I have proved the following stronger result: For every integer $n \geq 3$ and every $\varepsilon > 0$, there is an integer $g(n, \varepsilon)$ such that every finite graph G with $\|G\| > \left(\frac{n-1}{2} + \varepsilon\right)|G|$ and $\tau(G) \geq g(n, \varepsilon)$ contains a $\dot{K}_{n+1}$.

for all $1 \le i < j \le m+1$ so that the graph K_F arising in this way becomes n-regular. By Theorem 2, every n-connected graph G with $\tau(G) \ge g(K_F)$ contains a $\dot{K}_F$. Then one can show that there are a, x_i-paths P_i for $i \in \mathbb{N}_n$ in G with $V(P_i) \cap V(P_j) = \{a\}$ for $i \ne j$, so that $V(P_i) \cap V(\dot{F}_i) = \{x_i\}$ for all $i \in \mathbb{N}_n$ (after renumbering the F_j), where x_i is a branch vertex of $\dot{F}_i \subseteq \dot{K}_F$, and $P_1, \ldots, P_n$ together meet $\dot{K}_F - \{x_1, \ldots, x_n\}$ in at most n further F_j with $j > n$ and "subdivided edges" adjacent to these F_j. This configuration can be extented to an $\dot{H} \subseteq \bigcup_i P_i \cup \dot{K}_F$ with the required property. ∎

We will now study openly disjoint paths with prescribed endvertices in graphs of large girth and suitable connectivity. We need the following lemma, the proof of which is easily derived from the proof of the corresponding results in [4] and [5].

Lemma 1. *If a $2n$-connected graph contains a subdivision of the complete bipartite graph $K_{2n,2n}$, then it is n-linked.*

Sketch of proof. Assume $\kappa(G) \ge 2n$ and let $B = B_1 \dot{\cup} B_2$ be the set of branch vertices of a subdivision $H \subseteq G$ of $K_{2n,2n}$, where B_1, B_2 correspond to the maximal independent vertex sets in $K_{2n,2n}$. For $B' := \{b, b'\} \in \mathcal{P}_2(B)$ with $|B' \cap B_i| = 1$ for $i = 1, 2, H(b, b')$ denotes the b, b'-path in $H - (B - B')$.

Consider $A = \{a_1, \ldots, a_{2n}\} \in \mathcal{P}_{2n}(V(G))$. We construct n disjoint a_i, a_{i+n}-paths for $i \in \mathbb{N}_n$. By Menger's theorem and the methods from [4] and [5], we can find $2n$ disjoint a_i, b_i-paths P_i for $i \in \mathbb{N}_{2n}$ with $V(P_i) \cap B = \{b_i\}$ and the following property: If $P := H(b, b')$ for $\{b, b'\} \in \mathcal{P}_2(B)$ with $|\{b, b'\} \cap B_j| = 1$ and $b' \notin B_0 := \{b_1, b_2, \ldots, b_{2n}\}$ and if there is an $i \in \mathbb{N}_{2n}$ with $P_i \cap P \ne \emptyset$, then $b_i = b$ and $P_i[p, b_i] = P[p, b]$ for all $p \in P_i \cap P$ (or the first on P_i).

If $\{b_i, b_{i+n}\} \subseteq B_1$ and $b \in B_2 - B_0$, then $P_i \cup H(b_i, b) \cup H(b, b_{i+n}) \cup P_{i+n}$ is disjoint to $\bigcup_{j \ne i, i+n} P_j$ and contains an a_i, a_{i+n}-path. If $b_i \in B_1$, $b_{i+n} \in B_2$ and $b \in B_2 - B_0, b' \in B_1 - B_0$, then $P_i \cup H(b_i, b) \cup H(b, b') \cup H(b', b_{i+n}) \cup P_{i+n}$ is disjoint to $\bigcup_{j \ne i, i+n} P_j$ and contains an a_i, a_{i+n}-path. Now it is easy to see that one can distribute in this way (some of) the $2n$ vertices of $B - B_0$ to the n pairs (b_i, b_{i+n}) to get n disjoint a_i, a_{i+n}- paths in $H \cup \bigcup_{j=1}^{2n} P_j$. ∎

Let H be a graph with $V(H) = \{b_1, \ldots, b_k\}$. We say, that *a graph G contains a subdivision of H with prescribed branch vertices*, if $|G| \ge k$ and for every sequence of distinct vertices $a_1, \ldots, a_k$ of G there is a subdivision $\dot{H} \subseteq G$, where a_i is the branch vertex of $\dot{H}$ corresponding to b_i for all $i \in \mathbb{N}_k$. For instance, G is n-linked, if it has a subdivision with prescribed branch vertices of a graph H consisting of n disjoint edges.

Theorem 3. *Every $(2||H||)$-connected graph G with $\tau(G) \ge g(K_{2||H||,2||H||})$ contains a subdivision with prescribed branch vertices of the graph H without isolated vertices.*

Proof. Let H be a graph with $V(H)=\{b_1,\dots,b_k\}$ and without isolated vertices, hence $||H||>0$. Consider a graph G with $\kappa(G)\geq 2||H||=:m$ and $\tau(G)\geq g(K_{m,m})$ and let $a_1,\dots,a_k$ be distinct vertices of G. We assume $||H||\geq 2$, as the case $||H||=1$ is obvious. By Theorem 2. there is a $\dot{K}_{m,m}\subseteq G$. For every $i\in\mathbb{N}_k$, split a_i into an independent set $A_i\neq\emptyset$ of $d_H(b_i)$ vertices, so getting $\overline{G}$ from G. Since, obviously, $\kappa(\overline{G})\geq m$ and $\dot{K}_{m,m}\subseteq\overline{G}$, the graph $\overline{G}$ is $\frac{m}{2}$- linked by lemma 1. For every $[b_i,b_j]\in E(H)$ we can choose $a_i'\in A_i$ and $a_j'\in A_j$, such that every vertex of $\bigcup_{\alpha=1}^k A_\alpha$ occurs in exactly one (unordered) pair. So the $\frac{m}{2}$ disjoint a_i',a_j'-paths in $\overline{G}$ for $[b_i,b_j]\in E(H)$ give a subdivision of H in G with branch vertex a_i corresponding to b_i for $i\in\mathbb{N}_k$, since every element of $\bigcup_{\alpha=1}^k A_\alpha$ is endvertex of these disjoint paths and for every $i\neq j$ we have chosen at most one pair a',a'' with $a'\in A_i$ and $a''\in A_j$. ∎

Corollary 1. *Every $2n$-connected graph G with $\tau(G)$ large enough is n-linked. For all $n\geq 2$ this statement is not true for $2n-1$ instead of $2n$.*

Proof. The first assertion follows immediately from Theorem 3. So we assume $n\geq 2$ and consider any positive integer m. Since every graph G with $\delta(G)\geq 4k$ contains a k-connected subgraph by [7], there is a $(2n-1)$-connected graph H with $\tau(H)\geq m$, which contains an $X\in\mathcal{P}_{2n-1}(V(H))$, such that $d_H(x,y)\geq\frac{m}{2}$ for all $x\neq y$ from X. Consider disjoint sets $A=\{a_1,\dots,a_n\}$, $B=\{b_1,\dots,b_n\}$ and C with $|C|=n-1$. For every choice $Y\in\mathcal{P}_{2n-1}(A\cup B\cup C)$ with $C\subseteq Y$ and $|Y\cap\{a_i,b_i\}|=1$ for $i\in\mathbb{N}_n$, we take a copy H_Y of H, where X_Y corresponds to X, and identify the vertices of X_Y in a bijective manner with the vertices of Y, so getting a graph G. Then $\tau(G)\geq m$, and $\kappa(G)=2n-1$ as easily checked, but there are no n disjoint a_i,b_i-paths for $i=1,\dots,n$ in G. ∎

Corollary 2. *Every $2\cdot\binom{n}{2}$-connected graph of sufficiently large girth contains a subdivision of K_n with prescribed branch vertices.*

There are $(\binom{n}{2}-1)$ -connected graphs G of arbitrarily large girth which have a separating set T with $|T|<\binom{n}{2}$ such that $G-T$ has n components (cf. the proof of Corollary 1). This shows that $(\binom{n}{2}-1)$-connectivity and large girth cannot ensure the existence of a subdivision of K_n with prescribed branch vertices for $n\geq 2$. But I do not believe that Corollary 2 is best possible. I conjecture that $\binom{n}{2}$*-connectivity suffices* (as for $n\leq 3$). It could be possible that the proof which I sketched for the case of one prescribed vertex at the beginning of this section is extentable to the general case.

References

[1] B. Bollobás and A. Thomason: Highly linked graphs, *Conbinatorica*, **16** (1996), 313–320.

[2] R. Diestel: *Graphentheorie,* Springer-Verlag, Berlin, Heidelberg, 1996.

[3] H. A. Jung: Zusammenzüge und Unterteilungen von Graphen, *Math. Nachr.,* **35** (1967), 241–267.

[4] H. A. Jung: Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.,* **187** (1970), 95–103.

[5] D. G. Larman and P. Mani: On the existence of certain configurations within graphs and the 1-skeletons of polytopes, *Proc. London Math. Soc.,* **20** (1970), 144–160.

[6] W. Mader: Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Ann.,* **174** (1967), 265–268.

[7] W. Mader: Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Uni. Hamburg,* **37** (1972), 86–97.

[8] W. Mader: An extremal problem for subdivisions of K_5^-, to appear in *J. Graph Theory*.

[9] H. Sachs: Regular graphs with given girth and restricted circuits, *J. London Math. Soc.,* **38** (1963), 423–429.

[10] C. Thomassen: 2-linked graphs, *Europ. J. Combinatorics,* **1** (1980), 371–378.

[11] C. Thomassen: Girth in graphs, *J. Combin. Theory Ser. B,* **35** (1983), 129–141.

[12] W. T. Tutte: *Connectivity in graphs,* Math. Exposition 15, Univ. of Toronto Press, Toronto Ont., Oxford Univ. Press, London, 1966.

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